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Almost principal ideals and tensor product of hyperlattices

A. Soltani Lashkenari and B. Davvaz

Abstract

In this paper, by considering the notion of congruences on hyperlattices we define almost principal ideals on hyperlattices. We investigate some properties and prove some results about them. Also, we define compatible functions on hyperlattices and investigate connection between these functions and almost principal ideals. Then, we define tensor product of two hyperlattices and present several properties such as completeness and distributivity on tensor product of hyperlattices.

1 Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures and first introduced by Marty [15]. Till now, the hyperstructures are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics [3, 4]. Hyperlattices were first studied by Konstantinidou and Mittas [12].

We mention here only some names of mathematicians who have worked in lattices and hyperlattices: J.C. Varlet, T. Nakano, J. Mittas, A. Kehagias, M. Konstantinidou, K. Serafimidis, V. Leoreanu, I.G. Rosenberg, S. Rasouli, B. Davvaz, G. Calugareanu, G. Radu, A.R. Ashrafi, for example see [1, 6, 7, 10, 11, 13, 14, 16, 17, 21, 22, 23, 24, 26, 27]. In [8] Jakubik studied several aspects of the theory of superlattices; in particular he defined congruences

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on hyperlattices and studied the properties of the resulting quotients. Also, the congruence of hyperlattices are studied in [9]. Compatible functions on distributive lattices are studied in [18, 19, 20]. In [2, 5, 25] tensor product of lattices are investigated. In this article, first by considering congruences on hyperlattices we define compatible functions on lattices and we investigate the connection between these functions and special ideals on hyperlattices which we defined almost principal ideals. Also, in the second section we define tensor product of two hyperlattices and we investigate concepts such as distributivity and completeness on tensor product of hyperlattices.

2 Basic definitions

A lattice is a partially ordered set L such that for any two elements x, y of L, $glb\{x, y\}$ and $lub\{x, y\}$ exist. If L is a lattice, then we define $x \vee y = glb\{x, y\}$ and $x \wedge y = lub\{x, y\}$. This definition is equivalent to the following definition. Let L be a non-empty set with two binary operations $\wedge$ and $\vee$. Let for all $a, b, c \in L$, the following conditions satisfied:

(1) $a \wedge a = a$ and $a \vee a = a$;
(2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
(4) $(a \wedge b) \vee a = a$ and $(a \vee b) \wedge a = a$;

Then, $(L, \vee, \wedge)$ is a lattice.

Join hyperlattice. Let L be a non-empty set, $\bigvee : L \times L \to \wp^*(L)$ be a hyperoperation, where $\wp^*(L)$ is the family of all non-empty subsets of L, and $\wedge : L \times L \to L$ be an operation. Then, $(L, \bigvee, \wedge)$ is a join hyperlattice if for all $x, y, z \in L$ the following conditions hold:

(1) $x \in x \bigvee x$ and $x = x \wedge x$;
(2) $x \bigvee (y \bigvee z) = (x \bigvee y) \bigvee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
(3) $x \bigvee y = y \bigvee x$ and $x \wedge y = y \wedge x$;
(4) $x \in x \wedge (x \bigvee y) \cap x \bigvee (x \wedge y)$.

Let $A, B \subseteq L$. Then, $A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$, $A \bigvee B = \bigcup_{a \in A, b \in B} a \bigvee b$.

Let $(L, \bigvee, \wedge)$ be a join hyperlattice. Accroding to [24], we say L is a strong join hyperlattice if for all $x, y \in L$, $y \in x \bigvee y$ implies that $x = x \wedge y$. We say that 0 is a zero element of L, if for all $x \in L$ we have $0 \leq x$ and 1 is a unit of L if for all $x \in L$, $x \leq 1$. We say L is bounded if L has 0 and 1. And y is a complement of x if $1 \in x \bigvee y$ and $0 = x \wedge y$. A complemented hyperlattice is a bounded hyperlattice which every element has a complement. We say L is distributive if for all $x, y, z \in L$, $x \wedge (y \bigvee z) = (x \wedge y) \bigvee (x \wedge z)$. And L

is s-distributive if $x\bigvee(y\wedge z)=(x\bigvee y)\wedge(x\bigvee z)$. Notice that in lattices, the concepts of distributivity and s-distributivity are equivalent but in hyperlattice this is not true.

Definition 2.1. Let $(L,\bigvee,\wedge)$ be a join hyperlattice and $I,F\subseteq L$. We call I is an ideal of L if: (1) for every $x,y\in I$, $x\bigvee y\subseteq I$; (2) $x\leq I$ implies $x\in I$. Also, F is a filter of L if: (1) for every $x,y\in F$, $x\wedge y\in F$; (2) $x\leq a$ such that $x\in F$ implies $a\in F$.

Notice that I is a prime ideal if for any $x,y\in L$, $x\wedge y\in I$ implies that $x\in I$ or $y\in I$ and for every $a\in L$ the set $I=\{x\in L\mid x\leq a\}$ is an ideal of L which is called principal ideal.

Definition 2.2. Let $(L,\bigvee,\wedge)$ be a join hyperlattice and θ be an equivalence relation on L. Then, θ is a congruence on L if for any $x,y,z\in L$ and $x\ \theta\ y$ we have $x\bigvee z\ \theta\ y\bigvee z$ and $x\wedge z\ \theta\ y\wedge z$.

In this case, we define binary multioperations $\curlyvee,\curlywedge$ on L/θ by putting

$$x_\theta\curlyvee y_\theta=(x\bigvee y)_\theta, x_\theta\curlywedge y_\theta=(x\wedge y)_\theta,$$

for each $x_\theta,y_\theta\in L/\theta$. We denote $(L/\theta,\curlyvee,\curlywedge)=L/\theta$.

Example 1. [8] Let $\mathbb{R}$ be the set of all reals with the natural linear order. Furthermore, let S be the set of all pairs (x,y) with $x,y\in\mathbb{R}$. For $(x_1,y_1),(x_2,y_2)\in S$ we put $(x_1,y_1)\leq(x_2,y_2)$ if either $(x_1,y_1)=(x_2,y_2)$ or $y_1<y_2$. Then, $(S,\leq)$ is a partially ordered set. We define binary multioperations $\curlyvee,\curlywedge$ on S as follows.

Let $a,b\in S$. We denote by $a\curlywedge b$ the set of all lower bounds of the set $\{a,b\}$. Next we put

$$a\curlyvee b=b\curlyvee a=\begin{cases} S & \text{if } a=b\\ S-\{a\} & \text{if } a<b\\ S-\{a,b\} & a,b \text{ are incomparable.}\end{cases}$$

Then, $(L,\curlyvee,\curlywedge)$ is a superlattice and we define for $(x,y),(x',y')\in S$,

$$(x,y)\rho(x',y')\ \Leftrightarrow\ x=x'.$$

We have ρ is a congruence on L.

3 Almost principal ideals and compatible functions

In this section, we define almost principal ideals and compatible functions on join hyperlattices and we investigate connection between them. Also, we prove some results about them.

Definition 3.1. Let $(L, \bigvee, \wedge)$ be a join hyperlattice and $I \subseteq L$. We call I is an almost principal ideal if the intersection of L with every principal ideal is a principal ideal.

Similarly, we say that the filter F is principal if the intersection of F with every principal filter of L is a principal filter of L.

Example 2. Every principal ideal is an almost principal ideal.

Example 3. Consider $H = \{a, b, c\}$ and the following tables:

$\bigvee$	a	b	c
a	a	b	c
b	b	b	H
c	c	H	c

$\wedge$	a	b	c
a	a	a	a
b	a	b	a
c	a	b	c

Then, $I = \{a, b\}$ is not almost principal ideal, since $I \cap (b] = \{b\}$ is not principal ideal.

Let I, F be almost principal ideal and almost principal filter of L, respectively. For any $x \in L$ we set $x_I = Max(I \cap \downarrow x), x^F = Min(F \cap \uparrow x)$. Consider the function $f_I : L \to L$ such that for any $x \in L$ we have $f_I(x) = x_I$. We call f_I is the projection function of almost principal ideal I. Also, we define $f^F(x) = x^F$ for every almost principal filter F.

Definition 3.2. Let $(L, \bigvee, \wedge)$ be a join hyperlattice. Function

$$f : \underbrace{L \times L \times \ldots \times L}_{n\ times} \to L$$

is a compatible function if for every congruence θ and $a_i \theta b_i$ for $i = 1, 2, \ldots, n$, we have

$$f(a_1, a_2, \ldots, a_n)\ \theta f(b_1, b_2, \ldots, b_n).$$

Proposition 3.3. *If $(L, \bigvee, \wedge)$ is a distributive join hyperlattice, then the functions f_I, f^F are compatible functions.*

Proof. Suppose that $a\theta b$. We prove $f_I(a)\ \theta f_I(b)$. If $u \in\downarrow a$, then $u \leq a$ and $u \wedge a = u$. We have $u \wedge a\ \theta\ u \wedge b \leq b \in\downarrow b$. Thus, $u\ \theta\ u \wedge b = v$ and there exists $v \in\downarrow b$ such that $u\theta v$. Therefore, $\downarrow a\ \theta \downarrow b$. Since I is almost principal ideal, there exist $u, v \in L$ such that $\downarrow a \cap I =\downarrow u, \downarrow b \cap I =\downarrow v$ and $\downarrow u\theta \downarrow v$. Since $u \in\downarrow u$, it follows that there exists $c \in\downarrow v$ such that $u\theta c$. Since $c \in\downarrow v$, it follows that $c \leq v$ and $c \wedge v \ \theta \ u \wedge v, c \wedge v = c$. Hence, $c \ \theta \ u \wedge v$ and $u \wedge v \ \theta \ u$. Similarly, $u \wedge v \ \theta \ v, u\theta v$. Therefore, $f_I(a)\theta f_I(b)$. □

We denote the set of all almost principal ideals of L by $\mathcal{I}(L)$ and the set of all almost principal filters by $\mathcal{F}(L)$.

Proposition 3.4. *Let $(L, \bigvee, \wedge)$ be a strong join hyperlattice and $\mathcal{I}(L)$ be the set of almost principal ideals of L. The following conditions hold:*

(1) L is an ideal of $\mathcal{I}(L)$ and a filter of $\mathcal{F}(L)$;

(2) L is convex in $\mathcal{F}(\mathcal{I}(L)$;

(3) There exists a canonical homomorphism $\mathcal{F}(\mathcal{I}(L) \to \mathcal{I}(\mathcal{F}(L)$.

Proof. (1) First, we prove L is a subhyperlattice of $\mathcal{I}(L)$. Consider $\varphi : L \hookrightarrow \mathcal{I}(L)$ such that for every $x \in L$ we have $\varphi(x) =\downarrow x$. Since $\downarrow x$ is almost principal ideal of L, it follows that L is a subhyperlattice of $\mathcal{I}(L)$. If J is an almost principal ideal of L and $x \in L$, then $x \wedge J = \{x \wedge y \mid y \in J\} \subseteq J \subseteq L$. Therefore, L is an ideal of $\mathcal{I}(L)$. Similarly, we can prove that L is a filter of $\mathcal{F}(L)$.

(2) We show that L is convex in $\mathcal{I}(L)$. Let $x, y \in L, J \in \mathcal{I}(L)$ and $x \leq J \leq y$. We have $J \subseteq L$. Thus, all ideals and filters are convex. Therefore, L is convex in $\mathcal{I}(L)$ and $\mathcal{I}(L)$ is convex in $\mathcal{F}(\mathcal{I}(L))$.

(3) Consider $\varphi : \mathcal{F}(\mathcal{I}(L) \to \mathcal{I}(\mathcal{F}(L)$ such that $\varphi(G) = \{F \in \mathcal{F}(L) \mid F \cap J \neq \emptyset \textit{ for every } J \in G\}$ and for every $x \in L$, we consider $G = \{I \in \mathcal{I}(L) \mid \downarrow x \subseteq I\}$. We define the order relation on G as $F_1 \leq F_2$ if and only if $F_2 \subseteq F_1$. We show that G is a filter of $\mathcal{I}(L)$. Let $I_1, I_2 \in G$. Hence, we obtain

$$\downarrow x \subseteq I_1 \cap I_2 = I_1 \wedge I_2 = \{a \wedge b \mid a \in I_1, b \in I_2\}.$$

Thus, $I_1 \wedge I_2 \in G$. Now, let $G \leq I' \in \mathcal{I}(L)$. We show that $I' \in G$. There exists $I \in \mathcal{I}(L)$ such that $\downarrow x \subseteq I \leq I'$. Let $u \in\downarrow x$. Thus, $u \leq x \in I$ and $u \in I \leq I'$. So, there exists $u' \in I'$ such that $u \leq u' \in I'$. Therefore, $u \wedge u' = u \in I'$ and $\downarrow x \subseteq I'$. Hence, G is a filter of $\mathcal{I}(L)$. □

Notice that if $(L, \bigvee, \wedge)$ is a join hyperlattice, $P \subseteq L$ is a prime ideal of L and θ is a congruence of L, then we define the congruence of prime ideal P as $\theta_P = (P \times P) \cup (L \setminus P) \times (L \setminus P)$.

Proposition 3.5. *Let $(L, \bigvee, \wedge)$ be a hyperlattice, P be a prime ideal and I be an almost principal ideal of L. Then, for every $x, y \in L \setminus P$ we have $(x_I, y_I) \in \theta_P$.*

Proof. Since $x\theta_P y$ and f_I is a compatible function on L, we have $f_I(x)\theta_P f_I(y)$ and $x_I\theta_p y_I$. $\square$

In [23] Rasouli and Davvaz proved that if L is a hyperlattice, $I \subseteq L$ is an ideal and $x \notin I$, then there exists $P \in Spec(L)$ such that $I \subseteq P$, $x \notin P$ and for every $x, y \in L$ with $x \neq y$ there exists a prime ideal of L containing exactly one of x or y. Now, by considering the above fact we prove the following result.

Theorem 3.6. *Let $(L, \bigvee, \wedge)$ be a distributive strong join hyperlattice and $f : L^n \to L$ be a compatible function on L. If $I_1, I_2, \dots I_n$ are almost principal ideals of L, then, $J = \{x \in L \mid x \leq f(x_{I_1}, x_{I_2}, \dots, x_{I_n})\}$ is an almost principal ideal of L.*

Proof. Suppose that $y \leq x \in J$. We show that $y \in J$. Suppose that $y \notin J$. Then, $y \nleq f(y_{I_1}, y_{I_2}, \dots, y_{I_n})$. So, by the above results there exists a prime ideal P such that $f(y_{I_1}, y_{I_2}, \dots, y_{I_n}) \in P, y \notin P$. Since $y \leq x$ and P is a prime ideal, it follows that $y \notin P, x \notin P$. Therefore, $x, y \in L \setminus P$. By 3.5 $(x_{I_i}, y_{I_i}) \in \theta_i$. Since $x \leq f(x_{I_1}, x_{I_2}, \dots, x_{I_n})$ and $x \notin P$, it follows that $f(x_{I_1}, x_{I_2}, \dots, x_{I_n}) \notin P$. Thus, we have

$$(f(x_{I_1}, x_{I_2}, \dots, x_{I_n}), f(y_{I_1}, y_{I_2}, \dots, y_{I_n})) \notin \theta_P.$$

Since $(x_{I_i}, y_{I_i}) \in \theta_i$ and f is compatible function, the recent relation is a contradiction. Therefore, $y \in J$. Now, we prove that the intersection of J with every principal ideal is a principal ideal. We claim $y \wedge f(y_{I_1}, y_{I_2}, \dots, y_{I_n}) = Max(J \cap \downarrow y)$. We have $z = y \wedge f(y_{I_1}, y_{I_2}, \dots, y_{I_n}) \leq y$. Let $z \notin J$. Thus, there exists prime ideal Q such that $f(z_{I_1}, z_{I_2}, \dots, z_{I_n}) \in Q, z \notin Q$. Since $y \leq z \in Q$, it follows that $y \notin Q$. Similarly, we obtain $f(y_{I_1}, y_{I_2}, \dots, y_{I_n}) \notin Q$. Since $(y_{I_i}, z_{I_i}) \in \theta_Q$, it follows that $(f(y_{I_1}, y_{I_2}, \dots, y_{I_n}), f(y_{I_1}, y_{I_2}, \dots, y_{I_n})) \in \theta_Q$. This is a contradiction. Thus, $z \in J$. Also, $z \in \downarrow y$ and so $z \in J \cap \downarrow y$. Now, let $t \in J \cap \downarrow y$ such that $t \nleq z, t \leq y$. Then, there exists a prime ideal R such that $t \notin R, z \in R$. Since $t \notin R$, it follows that $y \notin R$. Moreover, $t \in J$. Therefore, $t \leq f(t_{I_1}, t_{I_2}, \dots, t_{I_n}) \notin R$. Since $z = y \wedge f(y_{I_1}, y_{I_2}, \dots, y_{I_n}) \in R$, it follows that $y \in R$ or $f(y_{I_1}, y_{I_2}, \dots, y_{I_n}) \in R$. Also, by compatibility of f and $(y_{I_i}, t_{I_i}) \in \theta_R$, we have $(f(y_{I_1}, y_{I_2}, \dots, y_{I_n}), f(t_{I_1}, t_{I_2}, \dots, t_{I_n})) \in \theta_R$ and this is a contradiction. Thus, for every $y \in L$, $Max(J \cap \downarrow y)$ exists. Now, we show that J is closed under $\bigvee$. Let $x, y \in J$. We have $x = x \wedge f(x_{I_1}, x_{I_2}, \dots, x_{I_n})$ and $y = y \wedge f(y_{I_1}, y_{I_2}, \dots, y_{I_n})$. Since $x, y \leq x \bigvee y$, it follows that $\downarrow x \cap J \subseteq \downarrow (x \bigvee y) \cap J$ and $\downarrow y \cap J \subseteq \downarrow (x \bigvee y) \cap J$. So, $Max(\downarrow x \cap J) \leq Max(\downarrow (x \bigvee y) \cap J)$ and $Max(\downarrow y \cap J) \leq Max(\downarrow (x \bigvee y) \cap J)$. Thus,we have

$$x, y \leq max(\downarrow (x \bigvee y) \cap J) = (x \bigvee y) \wedge ((f(x \bigvee y)_{I_1}, \dots, f(x \bigvee y)_{I_n}).$$

By the distributivity of L, $x \bigvee y \leq (f(x \bigvee y)_{I_1}, f(x \bigvee y)_{I_2}, \dots, f(x \bigvee y)_{I_n})$. Hence, $x \bigvee y \subseteq J$ and J is almost principal ideal. $\square$

Now, let $\bar{f} : (\mathcal{I}(L))^n \to \mathcal{I}(L)$ with $\bar{f}(I_1, I_2, \ldots, I_n) = \{x \in L \mid x \leq f(x_{I_1}, x_{I_2}, \ldots, x_{I_n})\}$.

Theorem 3.7. *Let $(L, \bigvee, \wedge)$ be a strong join hyperlattice and f be a compatible function on L. Then, $\bar{f}$ is a compatible function on $\mathcal{I}(L)$.*

Proof. Suppose that $\bar{f}$ is not compatible. Then, there exist $I_1, I_2, \ldots, I_n$, $J_1, J_2, \ldots, J_n$ such that $(I_i, J_i) \in \theta_P$. Without loss the generality, suppose that $\bar{f}(I_1, I_2, \ldots, I_n) \in P$ and $\bar{f}(J_1, J_2, \ldots, J_n) \notin P$. Set $M = \{i \in \{1, 2, \ldots, n\} \mid I_i \in P\}$. Thus, we obtain

$$\begin{aligned}&\bar{f}(J_1, J_2, \ldots, J_n) \wedge \wedge_{i \notin M} I_i \wedge \wedge_{i \notin M} J_i \\ &\nleq \bar{f}(I_1, I_2, \ldots, I_n) \bigvee (\bigvee_{i \in M} I_i) \bigvee (\bigvee_{i \in M} J_i).\end{aligned}$$

We have $\bar{f}(J_1, J_2, \ldots, J_n) \wedge \wedge_{i \notin M} I_i \wedge \wedge_{i \notin M} J_i \in P$. Since P is a prime ideal, it follows that $\wedge_{i \notin M} J_i \in P$ or $\wedge_{i \notin M} I_i \in P$. Thus, we have$I_i \in P$ or $J_i \in P$ and this is a contradiction. So, by the definition there exists $y \in \bar{f}(J_1, J_2, \ldots, J_n)$ such that $y \nleq \bar{f}(I_1, I_2, \ldots, I_n) \bigvee \bigvee_{i \in M} I_i \bigvee \bigvee_{i \in M} y_i$. There exists a prime ideal Q such that $y \notin Q$ and $f(y_{I_1}, y_{I_2}, \ldots, y_{I_n}) \bigvee \bigvee_{i \in M} I_i \bigvee \bigvee_{i \in M} y_i \in Q$. So, we have $f(y_{I_1}, y_{I_2}, \ldots, y_{I_n}) \in Q$. But $y \in \bar{f}(J_1, J_2, \ldots, J_n)$. Therefore, $y \leq f(y_{J_1}, y_{J_2}, \ldots, y_{J_n})$ and $f(y_{J_1}, y_{J_2}, \ldots, y_{J_n}) \notin Q$. This implies that

$$(f(y_{I_1}, y_{I_2}, \ldots, y_{I_n}), f(y_{J_1}, y_{J_2}, \ldots, y_{J_n})) \notin \theta_Q.$$

We have $y_{I_i} = Max(I_i \cap \downarrow y)$ and $I_i \leq \bigvee_{i \in M} I_i \subseteq Q$. Then, for every $i \in M, I_i \subseteq Q$ and for every $j \in M, J_i \subseteq Q$. Thus, we obtain $(y_{I_i}, y_{J_i}) \in \theta_Q$ and $(f(y_{I_1}, \ldots, y_{I_n}), f(y_{J_1}, \ldots, y_{J_n})) \in \theta_Q$. This is a contradiction and so $\bar{f}$ is compatible function on $\mathcal{I}(L)$. $\square$

4 Tensor product of two hyperlattices

In this section, we introduce tensor product of two hyperlattices and investigate some related properties.

Definition 4.1. Let L_1, L_2 be two bounded join hyperlattices. We demonstrate tensor product of two hyperlattices by $L_1 \otimes L_2$, where

$$L_1 \otimes L_2 = \{T \mid T = \bigvee_\theta L_b^a = \bigvee\{a \otimes b \mid (a, b) \in \theta\},$$

$$L_b^a(x) = \begin{cases} 1 & \text{if } x = 0 \\ b & \text{if } 0 < x \leq a \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\theta = \{(a,b) \mid T(a) \geqslant b\}$ and $a \otimes b = L_b^a(x) \in L_1 \otimes L_2$. Consider $t_1 : L_1 \to L_1 \otimes L_2$ with $t_1(a) = L_1^a = a \otimes 1$ for every $a \in L_1$ and $t_2 : L_2 \to L_1 \otimes L_2$ with $t_2(b) = L_b^1 = 1 \otimes b$ for every $b \in L_2$. We demonstrate $L_b^a(x) = a \otimes b$. Thus, we have

$$\begin{aligned} L_1^a(x) \wedge L_b^1(x) &= \begin{cases} 1 & x = 0 \\ 1 \wedge b = b & 0 < x \leq a \\ b \wedge 0 = 0 & \text{otherwise} \end{cases} \\ &= L_b^a(x) = a \otimes b. \end{aligned}$$

If L_1, L_2 are s-good hyperlattices, then

$$\begin{aligned} L_1^a(x) \bigvee L_b^1(x) &= \begin{cases} 1 \bigvee 1 & x = 0 \\ 1 \bigvee b \wedge b = b & 0 < x \leq a \\ b \bigvee 0 = 0 & \text{otherwise} \end{cases} \\ &= E_b^a(x). \end{aligned}$$

Notice that in $L_1 \otimes L_2$, set $E_b^a = a \bigvee b$ and $L_b^a = a \wedge b$.

Proposition 4.2. *Let L_1, L_2 be two bounded s-good join hyperlattices and for every $x \in L_2$ we have $x \bigvee 1 = x$. Also, for every $a_1, a_2 \in L_1$ and $b_1, b_2 \in L_2$, $a_1 \wedge b_1 \leq a_2 \bigvee b_2$ in $L_1 \otimes L_2$. Then, $a_1 \leq a_2$ or $b_1 \leq b_2$.*

Proof. Suppose that $a_1 \nleq a_2$. Then, we have $a_1 \wedge b_1 = L_{b_1}^{a_1}(a_1) = b_1 \leq E_{b_2}^{a_2}(a_1) = 1 \bigvee b_2 = b_2$. $\square$

Definition 4.3. Let L_1, L_2 be two bounded complete join hyperlattices. A complete distributive join hyperlattice D is the free product of L_1, L_2 if there exists homomorphism $\varepsilon_1 : L_1 \to D$ and $\varepsilon_2 : L_2 \to D$ such that $\varepsilon_2 \circ T = \varepsilon_1$, where $T : L_1 \to L_2$ is a homomorphism of hyperlattices and for every complete distributive hyperlattice K and homomorphisms $f_1 : L_1 \to K$ and $f_2 : L_2 \to K$, there exists a homomorphism $f : D \to K$ such that $f \circ \varepsilon_1 = f_1, f \circ \varepsilon_2 = f_2$.

Theorem 4.4. *Let L_1, L_2 be two bounded complete distributive join hyperlattice and for every $x \in L_1$ or L_2 we have $x \bigvee 1 = x$. Then, $L_1 \otimes L_2$ is complete distributive free product of hyperlattices L_1 and L_2.*

Proof. First, we show that for every $a, b, c \in L_1 \otimes L_2$, $(a \bigvee b) \wedge c = (a \wedge c) \bigvee (b \wedge c)$. In other hand, we show that $L_c^{E_a^b} = E_{L_c^b}^{L_c^a}$. It is easy to see that the recent relation holds. Now, suppose that $S \subseteq L_1 \otimes L_2$. So, we have

$$\begin{aligned} S^u &\subseteq \{b' \in L_2 \mid \forall b \in L_2,\ b \wedge b' = b\}, \\ S^l &\subseteq \{a' \in L_1 \mid \forall a \in L_1, a \wedge a' = a\}. \end{aligned}$$

Hence, S^u and S^l have the least element. Now, consider $t_1 : L_1 \to L_1 \otimes L_2$ with $t_1(a) = a \otimes 1 = L_1^a$ and $t_2 : L_2 \to L_1 \otimes L_2$ with $t_2(b) = 1 \otimes b = L_b^1$. Let K be an arbitrary complete distributive hyperlattice and $f_1 : L_1 \to K, f_2 : L_2 \to K$ be homomorphisms. Consider the homomorphism $\hat{f} : L_1 \times L_2 \to K$ with $\hat{f}(a,b) = f_1(a) \wedge f_2(b)$. We expand $\hat{f}$ to $f : L_1 \otimes L_2 \to K$. Thus, for every $T \in L_1 \otimes L_2$, we have

$$f(T) = f(\bigvee_{i\in I} L_{b_i}^{a_i}) = \bigvee_{i\in I} f(a_i, b_i) = \bigvee_{i\in I}(f_1(a_i) \wedge f_2(b_i)).$$

In particular, $f(L_1^a) = f_1(a) \wedge f_2(1) = f_1(a) \wedge f_1(1 \otimes 1) = f_1(a \wedge 1) = f_1(a)$. Also, $f \circ t_1(a) = f(a \otimes 1) = f(L_1^a) = f_1(a)$ and $f \circ t_2(b) = f(1 \otimes b) = f_1(1) \wedge f_2(b) = f_2(b)$. Now, we show that f is a homomorphism of hyperlattices. We have

$$\begin{aligned} f(T) &= \bigvee_{i\in I} f_1(a_i) \wedge f_2(b_i) \\ &= \wedge_{2^I}(\bigvee_J f_1(a_i) \bigvee_{I-J} f_2(b_i) \\ &= \wedge_{2^I}\{f_1(\bigvee_J a_i) \bigvee f_2(\bigvee_{I-J} b_i)\} \\ &= \wedge_{2^I}\{f_1(a_j) \bigvee f_2(b_j))\}. \end{aligned}$$

Also, f preserves $\wedge$ and $L_1 \otimes L_2$ is the free product of hyperlattices. □

Definition 4.5. Let $(L, \bigvee, \wedge)$ be a bounded complete distributive hyperlattice and $A' \subseteq L$. We call A' is independent if for every $a \in A'$, we have $0 \in a \wedge \bigvee_{a' \in A' - a'} a'$.

Proposition 4.6. *Let L_1, L_2 be complete hyperlattices. If $\{a_i \mid i \in I\}$ and $\{b_j \mid j \in J\}$ are independent subsets of L_1 and L_2, then $U = \{a_i \otimes b_j \mid (i,j) \in I \times J\}$ is independent subset of $L_1 \otimes L_2$.*

Proof. Suppose that U is not independent. Then, we have

$$0 \notin a_i \otimes b_j \wedge \bigvee\{a_i \otimes b_j \mid (i,j) \in I \times J - (i', j')\}.$$

Thus, there exists $T \in L_1 \otimes L_2$ such that $0 \notin (L_{b'_j}^{a'_i} \wedge T)(x)$. Therefore, for $0 < x \leq a'_i$, we obtain $0 \notin b'_j \wedge T(x)$ and $0 \notin b'_j \wedge \bigvee_{j \in J - j'} b_j$. Thus, $\{b_j \mid j \in J\}$ is an independent subset of L_2 and this is a contradiction. □

Theorem 4.7. *Let L_1, L_2 be two bounded distributive strong s-good join hyperlattices such that for every $b \in L_2$, $1 \in 1 \bigvee b$ and $L_1 \otimes L_2$ is a distributive hyperlattice.*

(1) If $a \in L_1, b \in L_2$ are meet-irreducible elements, then E_b^a is meet-irreducible in $L_1 \otimes L_2$;

(2) If $T \in L_1 \otimes L_2$ is meet-irreducible, then for every $x \in L_1$, $T(x) \in L_2$ is meet-irreducible.

Proof. (1) Suppose that $T_1 \wedge T_2 = E^a_b$. If $x = 0$, then $T_1(x) \wedge T_2(x) = 1 \bigvee 1$. Thus, $T_1(x) = T_2(x) = 1$. If $0 < x \leq a$, then $E^a_b(x) = 1 \bigvee b$. Therefore, $T_1 \wedge T_2 = 1 \bigvee b$. Since $1 \bigvee b \leq T_1(x), T_2(x)$ and $T_1(x), T_2(x) \leq 1 \in 1 \bigvee b$, it follows that $T_1(x) = T_2(x) = 1 \bigvee b$. If $x \nleq a$, then $T_1 \wedge T_2 = b$. Since b is meet-irreducible, it follows that $T_1 = b$ or $T_2 = b$.

(2) Suppose that $T(x) = m_1 \wedge m_2$, $T(x) \neq m_1, m_2$, $T_1 = T \bigvee L^a_{m_1}$ and $T_2 = T \bigvee L^a_{m_2}$. Then, we have $T_1 \wedge T_2 = T \bigvee (L^a_{m_1} \wedge L^a_{m_2})$. Let $x \leq m_1, m_2$. We have $T(x) = T(x) \bigvee (m_1 \wedge m_2)$. Thus, by the distributivity we obtain $T = T_1 \wedge T_2$ and since T is meet-irreducible, we have $T = T_1$ or $T = T_2$. For every $x \leq a$, we have $T_1 = (m_1 \wedge m_2) \bigvee m_1 \neq m_1 \wedge m_2$ and $T_2 = (m_1 \wedge m_2) \bigvee m_2 \neq m_1 \wedge m_2$. This is a contradiction with meet-irreducibility of T. Thus, $T(x) = m_1$ or $T(x) = m_2$. $\square$

Definition 4.8. Let L_1, L_2 be two join hyperlattices and $\theta \subseteq L_1 \times L_2$. We call that θ is a G-ideal if θ is an ideal and $(a_i, b_j) \subseteq \theta$ implies that $(\wedge_{i\in I} a_i, \bigvee_{j\in J} b_j) \subseteq \theta$ and $(\bigvee_{i\in I} a_i, \wedge_{j\in J} b_j) \subseteq \theta$.

Theorem 4.9. *L_1, L_2 are two complete distributive strong infinite join hyperlattices if and only if $L_1 \otimes L_2$ is a complete distributive infinite join hyperlattice. (Notice that in if part of theorem L_1, L_2 should be ordered and order $\leq, \ll$ should be coincide).*

Proof. We make a one to one correspondence between $T \in L_1 \otimes L_2$ and $\theta = \{(a, b) \mid T(a) \geqslant b\}$. By the definition of $L_1 \otimes L_2$, the construct of $\bigvee T_i$ is equivalent to the construct of $\cup\theta_i = \cup\sigma(T_i)$. We use the induction for this construction. Put $\theta^{(0)} = \cup\sigma(T_i)$ and for each ordinal number ε such that $\varepsilon = \tau + 1$ put

$$\theta^{\varepsilon} = \{(x, y) \mid (x, y) \leq (\bigvee x_i, \wedge y_i) \text{ or } (x, y) \leq (\wedge x_i, \bigvee y_i), (x_i, y_i) \subseteq \theta^{\tau}\}.$$

When ε is a limit ordinal, we put $\theta^{\varepsilon} = \cup_{\tau<\varepsilon}\theta^{\tau}$. Suppose that ε_0 is the first ordinal number such that $\theta^{\varepsilon_0} = \theta^{\varepsilon_0+1}$. Then, we have $\theta = \sigma(\bigvee T_i) =< \cup\sigma(T_i) >$. We show that θ is a G-ideal of $L_1 \times L_2$. Let $(x_1, y_1), (x_2, y_2) \in \theta$. Thus, there exist $\varepsilon_1, \varepsilon_2 \leq \varepsilon_0$ such that $(x_1, y_1) \in \theta^{\varepsilon_1}, (x_2, y_2) \in \theta^{\varepsilon_2}$. We show that $(x_1 \bigvee x_2, y_1 \bigvee y_2) \in \theta^{\varepsilon}$. This relation holds by the definition. Also, the second condition of ideals holds. If $(a_i, b_j) \subseteq \theta$, then $(a_i, b_j) \in \theta^{\varepsilon}$. Thus, $(a_i, b_j) \leq (\bigvee x_i, \wedge y_i)$ or $(a_i, b_j) \leq (\wedge x_i, \bigvee y_i)$. If $a_i \leq \bigvee x_i, b_i \leq \wedge y_i$, then we have $(\bigvee a_i, \wedge b_j) \subseteq \theta^{\varepsilon}$. Therefore, θ is a G-ideal. Since $\theta^0 = \cup\sigma(T_i)$, by the definition, we have $\theta =< \cup\sigma(T_i) >$ is a G-ideal. Now, let $T, T_i \in L_1 \otimes L_2$. We show that $T \wedge \bigvee_{i\in I} T_i = \bigvee_{i\in I}(T \wedge T_i)$. Since $T_i \leq \bigvee T_i$, $\leq$ is an order and by the coincidence of two orders, we have $\bigvee(T \wedge T_i) \leq T \wedge (\bigvee_{i\in I} T_i)$. For the converse, let $(x, y) \in \sigma(T) \cap \sigma(\bigvee T_i)$ such that $\sigma(T)$ is G-ideal correspondence

T. Thus, $(x,y) \in \sigma(T)$ and we obtain

$$(x,y) \in \sigma(\bigvee T_i) = \{(x,y) \mid (x,y) \in \theta^{\varepsilon}\}$$

such that $\varepsilon \leq \varepsilon_0$. We prove by the induction on ε. We show that $(x,y) \in \sigma(\bigvee(T \wedge T_i))$. If $\varepsilon = 0$, then $(x,y) \in \sigma(T_i)$ and $(x,y) \in \sigma(T)$. Thus, $(x,y) \in \sigma(T) \wedge \sigma(T_i)$ and $(x,y) \in \bigvee(\sigma(T) \wedge \sigma(T_i))$. Now, let for $\varepsilon' < \varepsilon$ results hold. If $\varepsilon = \tau + 1$, then $(x,y) \leq (\bigvee x_i, \wedge y_i)$ or $(x,y) \leq (\wedge x_i, \bigvee y_i)$ for $(x_i, y_i) \subseteq \theta^{\tau}$. Since L_1, L_2 are distributive, it follows that $(x,y) = (\bigvee x_i, \wedge y_i)$ and in the second case $(x,y) = (\wedge x_i, \bigvee y_i)$. We have $(x_i \wedge x, y_i \wedge y) \leq (x_i, y_i) \in \theta^{\tau}$ such that $\tau < \varepsilon = \tau + 1$ and $(x_i \wedge x, y_i \wedge y) \leq (x,y) \in \sigma(T)$. Since $\sigma(T)$ is a G-ideal, it follows that $(x_i \wedge x, y_i \wedge y) \in \sigma(T)$. By hypothesist, $(x_i \wedge x, y_i \wedge y) \in \sigma(\bigvee(T \wedge T_i))$. Since $\sigma(\bigvee(T \wedge T_i))$ is a G-ideal, it follows that $(\bigvee x_i, \wedge y_i) \in \sigma(\bigvee(T \wedge T_i))$ and $(\wedge x_i, \bigvee y_i) \in \sigma(\bigvee(T \wedge T_i))$. By properties of G-ideals, we have $(x,y) \in \sigma(\bigvee(T \wedge T_i))$. If ε is a limit ordinal number, by the definition of θ^{ϵ} we have $(x,y) \in \sigma(\bigvee(T \wedge T_i))$. Therefore, $\sigma(T) \cap \sigma(\bigvee T_i) \subseteq \sigma(\bigcap(T \wedge T_i))$ and $L_1 \otimes L_2$ is distributive. $\square$

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A. SOLTANI LASHKENARI,
Department of Mathematics,
Yazd University,
Yazd, Iran.

B. DAVVAZ,
Department of Mathematics,
Yazd University,
Yazd, Iran.
Email: davvaz@yazd.ac.ir, bdavvaz@yahoo.com